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### A Newton-like method for error analysis

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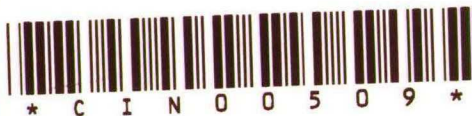
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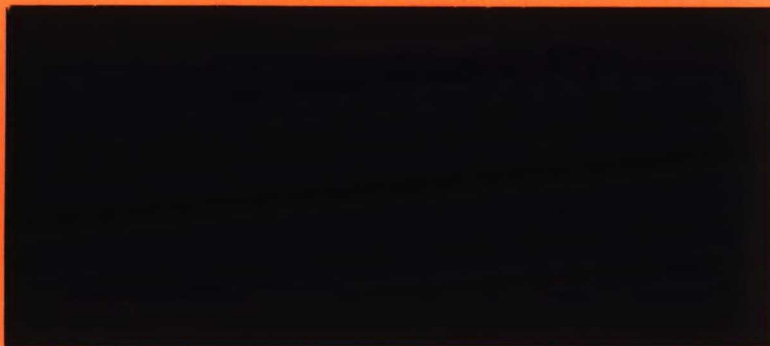
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## RESEARCH MEMORANDUM



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SUBFACULTEIT DER ECONOMETRIE

A NEWTON-LIKE METHOD FOR ERROR ANALYSIS

APPLIED TO LINEAR CONTINUOUS SYSTEMS AND EIGEN-  
PROBLEMS

M.H.C. Paardekooper

december 1981



## A NEWTON-LIKE METHOD FOR ERROR ANALYSIS

### APPLIED TO LINEAR CONTINUOUS SYSTEMS AND EIGENPROBLEMS

#### ABSTRACT

This paper discusses bounds for the distance of a point  $a$  in Hilbert space  $X$  to a manifold  $S = \{x \in X | F(x) = 0\}$ , where  $f$  is a Frechet differentiable mapping into a Hilbert space  $Y$ . The Newton-Kantorovich method handled in the normal space at  $a$  of the manifold  $\tilde{S} = \{x \in X | F(x) = F(a)\}$ , gives a realistic upperbound of this distance. In the usual Kantorovich conditions the Lipschitz continuity of  $Df$  effects  $S$  to be locally in a set with positive distance to  $a$ ; this leads to a lower bound of the distance  $d(a, S)$ . This approach leads to a method to distribute, in an "optimal" way the errors in a linear dynamical system among input, output and structural variables. A posteriori error analysis in eigenproblems illustrates also this approach.

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## 1. INTRODUCTION

In applied mathematics it is of frequent occurrence to have only an approximate solution of an equation. Only approximately the data satisfy the model equations, despite accuracy the numerical solutions have been applicated with errors, the structural parameters supposed to be constant are as well approximately known,...

For this reason it is important to have at disposal a manageable technique to determine sharp bounds for these errors. Both upper and lower bounds for the errors are of importance in the evaluation and diagnostics of models or to assess a numerical algorithm.

In the late forties Kantorovich proved the convergence of Newton's method for solving equations (, without assuming previously the existence of a solution,) under meanwhile standard assumptions [9], [13]. As an impressive result he also obtained an upperbound of the distance of starting value  $x_0$  to the locally unique solution  $x_\infty$ , being the limit of the Newton sequence. Lower bounds of this distance are given by Gragg and Tapia [8]. In the Newton-Kantorovich theory one considers the convergence of the Newton sequence to a zero of the sufficiently smooth function

$$f : X \supset W \rightarrow Y \quad (1.1)$$

where  $W$  is some ball around  $a = x_0$  in a Banach space  $X$  and the inverse of derivative  $Df(0)$  maps Banach space  $Y$  onto  $X$ . At the moment we remind and emphasize the Kantorovich-Gragg-Tapia bound for  $x_\infty - x_0$ :

$$m \|Df(x_0)^{-1} f(x_0)\| \leq \|x_\infty - x_0\| \leq M \|Df(x_0)^{-1} f(x_0)\|, \quad (1.2)$$

where, according to [13]  $M = (1 - \sqrt{1-2\kappa})/\kappa$  and according to [8]  $m = (\sqrt{1+2\kappa} - 1)/\kappa$ . Here  $\kappa = L\mu^{-1} \|Df(x_0)^{-1} f(x_0)\| < \frac{1}{2}$  where  $L$  is a Lipschitz constant of  $Df$  and  $\|Df(x_0)^{-1}\| \leq \frac{1}{\mu}$ .

In section two of this memorandum we discuss a generalization of the Newton-Kantorovich theorem. We consider a function  $f$  as given in (1.1), but now Frechet derivative  $Df(0): X \rightarrow Y$  is surjective. A lower and an upper bound is derived for the distance  $d(a, S)$  of  $a \in W$  to the manifold

$S = \{x \in W \mid f(x) = 0\}$ . As concerns  $f$  we assume derivative  $Df: W \rightarrow L(X, Y)$  to be Lipschitz continuous. Let be  $\varphi$  the restriction of  $f$  to the normal space  $N_2$  at  $a \in W$  to the level surface  $\{x \in W \mid f(x) = f(a)\}$ .

Provided  $\varphi$  satisfies the usual Kantorovich conditions, the intersection of  $W$  and  $N_2$  contains a zero, say  $z$ , of  $f: z \in S$ . So  $\|a - z\|$  is an upper-bound of  $d(a, S)$ . Now the Lipschitz continuity of  $Df$  has as consequence that  $S$  around  $z$  fluctuates inside a set that can be described by a simple inequality; this inequality gives a lowerbound of  $d(a, S)$ .

In practical situations many times the problem under consideration rises in the following way. Assume in some model, with given parameters  $a \in \mathcal{P}$ , some variables  $v \in \mathcal{V}$  have to satisfy the equation  $G(y; a) = 0$ . As a result of errors  $G(v, a) \neq 0$ . Then the question rises to find bounds for the perturbation  $e \in \mathcal{P}$  and  $d \in \mathcal{V}$  such that  $f(d, e) = 0$ , where

$$f(d, e) = G(v + d; a + e).$$

In previous papers [14], [15] we have found realistic, and rather usable upperbounds for these feasible perturbations  $(d, e) \in \mathcal{V} \times \mathcal{P}$  in the least squares problem and in the discrete linear model.

In section three we analyse the magnitude of sufficient perturbations  $(\alpha, \beta, \rho, \xi, \eta)$  in the continuous linear model

$$\begin{aligned} f(\alpha, \beta, \rho, \xi, \eta)(t) &:= \dot{y} + \dot{\eta} - (a + \alpha(t))(y + \eta(t)) - \\ & (b + \beta(t))(x + \xi(t)) - \rho(t) = 0 \end{aligned} \quad (1.3)$$

with given  $a, b \in \mathbb{R}$ ,  $x \in C[0, 1]$  and  $y \in C^1[0, 1]$ .

The non-linearity of  $f$  makes it necessary to use supremum norms and the contraction theorem in, say, in  $(Df(0, 0, 0, 0, 0))^*$ .

Hence only an upperbound of  $d(0, S)$  has been obtained; for the rest, that upperbound is sharp for small  $r := f(0, 0, 0, 0, 0)$ .

In section four we consider the eigenproblem as an area of application of the mentioned approach. There we describe, in addition to the error bounds, also a method for the improvement of an approximated eigenpair  $(\lambda, x)$ : a zero  $(\delta, h)$  of the function  $A(x + h) - (\lambda + \delta)(x + h)$  is a solution of an underdetermined system [2], [16]. But given that  $(\lambda, x)$  also the question

tries to construct an optimal perturbation  $(H, h)$  such that  $(A + H)(x + h) - \lambda(x + h) = 0$ . Numerical results for this problem demonstrate that if  $\lambda$  is an ill considered eigenvalue than a very small  $H$  suffices to fulfill the equation  $(A + H)(x + h) = \lambda(x + h)$ .

## 2. BOUNDS FOR THE DISTANCE OF A MANIFOLD IN A HILBERT SPACE TO A NEARBY POINT.

In this section we describe a method to determine an upper and a lower bound for the distance of a given point  $a$  in a Hilbert space  $X$  to a manifold  $S$ .

Let  $S$  be a manifold imbedded in  $X$ , defined as the preimage of zero in  $Y$  of a surjective mapping  $f$  of  $X$  into Hilbert space  $Y$ :

$$S := \{x \in X \mid f(x) = 0\} \quad (2.1)$$

In  $X$  the distance of  $a$  to  $S$  is defined by

$$d(a, S) = \inf\{\|x - a\| \mid x \in S\}.$$

The bounds for  $d(a, S)$  are derived under the assumptions that  $f$  is a mapping with Lipschitz continuous derivatives in some ball around  $a$  and that  $f(a)$ ,  $Df(a)$  and this Lipschitz constant for  $Df$  fit into the wellknown Kantorovich conditions. So the point  $a$  is near to  $S$  as can be described thanks these conditions.

The bounds for  $d(a, S)$  are obtained from the restriction of  $f$  to the linear manifold  $a + N_2$ , where  $N_2$  is the orthogonal complement of the nullspace  $N_1$  of  $Df(a)$ . Since  $N_2 = \text{im}(Df(a)^*)$ , we find for that restriction  $f|_{N_2}$ :

$$x \mapsto f(a + x), \quad x \in N_2 = \{Df(a)^* y \mid y \in Y\}.$$

If  $f(a + Df(a)^* y) = 0$  then

$$d(a, S) \leq \|Df(a)^* y\|$$

In order to prepare and structure the proof of theorem 2.1 we start to formulate some lemmata.

**LEMMA 2.1.** Let  $N_1$  be a closed subspace of Hilbertspace  $X$  and  $N_2 = N_1^\perp$ . Let  $\alpha, \beta$  be positive reals and  $w = w_1 + w_2 \in B(0, \beta)$  with  $w_i \in N_i$  ( $i = 1, 2$ )



and  $w_1 \neq 0$ . Let be

$$K := \{(1 - \|w_1\|^{-1}\tau)w_1 + x_2 \in X \mid \tau \in [0, \|w_1\|],$$

$$x_2 \in N_2 \cap B(w_2, \alpha\tau)\} \quad (2.2)$$

a subset of a convex cone in  $X$  with top  $w$  and let

$$W := \{x_1 + x_2 \in X \mid \alpha\|x_1\| + \|x_2\| < \beta, x_i \in N_i, i = 1, 2\} \quad (2.3)$$

Then  $K \subset B(0, \beta)$  if and only if  $w \in W$ .

PROOF. (i) Assume  $w \in W$ . If  $x = x_1 + x_2 \in K$ , then there exists a  $\tau \in [0, \|w_1\|]$  such that  $x_1 = (1 - \|w_1\|^{-1}\tau)w_1$  and  $x_2 = w_2 + \varepsilon\alpha\tau v$  with  $\varepsilon \in [0, 1)$  and  $v$  a unit vector in  $N_2$ .

Hence we investigate

$$\|x\|^2 = \|x_1\|^2 + \|x_2\|^2 = (\|w_1\| - \tau)^2 + \|w_2 + \varepsilon\alpha\tau v\|^2$$

$$\leq (\|w_1\| - \tau)^2 + (\|w_2\| + \alpha\tau)^2 =: g(\tau), \quad 0 \leq \tau \leq \|w_1\|.$$

Now on the interval  $[0, \|w_1\|]$  the quadratic function  $g$  attains its maximum in one of the endpoints  $0, \|w_1\|$ . For  $\tau = 0$  we obtain  $g(0) = \|w_1\|^2 + \|w_2\|^2 < \beta^2$  since  $w \in B(0, \beta)$  as given. For  $\tau = \|w_1\|$  we obtain  $g(\|w_1\|) = (\alpha\|w_1\| + \|w_2\|)^2 < \beta^2$  since  $w \in W$  as assumed. So  $x \in B(0, \beta)$  since  $0 < g(\tau) < \beta^2$  for each  $\tau \in [0, \|w_1\|]$ .

(ii) Conversely, suppose  $w \notin W$ , i.e.  $\alpha\|w_1\| + \|w_2\| \geq \beta$ . Now consider  $x := (1 - \|w_1\|^{-1}t)w_1 + x_2 \in K$  with  $t = \|w_1\|$  and

$$x_2 = \begin{cases} \alpha\|w_1\|v, & \text{where } v \in M_2, \|v\| = 1, \text{ if } w_2 = 0 \\ w_2 + \alpha \frac{\|w_1\|}{\|w_2\|} w_2 & , \text{ if } w_2 \neq 0. \end{cases}$$

Then  $x = x_2$  and thus  $\|x\| = \alpha\|w_1\| + \|w_2\| \geq \beta$ . So  $x \notin B(0, \beta)$ . □



The spherical symmetric forms in the next lemma make its proof to be a simple exercise.

LEMMA 2.2. Let  $w \in X$  and  $W$  as in lemma 2.1. The distance  $d(0, W^C)$  equals  $\beta(1 + \alpha^2)^{\frac{1}{2}}$ . □

In the proof of the main theorem 2.1 frequently the decomposition  $X = N_1 \oplus N_2$ , with  $N_2 = N_1^\perp$ , is used. Accordingly vectors  $x, y$  are decomposed:  $x = x_1 + x_2, y = y_1 + y_2$  with  $x_i, y_i \in N_i, i = 1, 2$ . As appears, the subspace  $N_1$  is the nullspace of the Frechet derivative at a given point  $a$  in the domain of a differentiable mapping.

THEOREM 2.1. Let  $X, Y$  be Hilbertspaces and  $f: X \supset B(a, r) \rightarrow Y$  ( $r > 0$ ) a Frechet differentiable mapping such that

- (i)  $A := Df(a) \in L(X, Y)$  is surjective and  $\|A^+\| \leq \lambda^{-1}$ , where  $A^+ \in L(X, Y)$  is the right inverse of  $A$ ;
- (ii)  $\|Df(x) - Df(y)\| \leq L\|x - y\|, x, y \in B(a, r)$ ;
- (iii)  $\|A^+ f(a)\| = \tilde{\gamma} \leq \gamma$ ;
- (iv)  $\kappa := L\gamma \lambda^{-1} < \frac{1}{2}$  and  $\rho := \lambda(1 - \sqrt{1-2\kappa})/L < r$ .

Then the equation  $f(x) = 0$  has a solution  $z$  in  $B(a, \rho) \cap N_2$  which is unique in  $B(a, \rho_2) \cap N_2$  where  $N_2$  is the orthogonal complement of  $N_1 := \ker(A)$  and

$$\rho_2 = \lambda(1 + \sqrt{1-2\kappa})/L. \quad (2.4)$$

The distance  $d(a, S)$ , where  $S = \{x \in B(a, r) \mid f(x) = 0\}$ , satisfies the inequalities

$$\rho_1 := \rho_3(1 + L^2 \rho_3^2 (\lambda - L\rho_3)^{-2})^{-\frac{1}{2}} < d(a, S) < \rho, \quad (2.5)$$

where

$$\rho_3 = \lambda(1 + \sqrt{1 - 2L\gamma\tilde{\lambda}^{-1}})/L \quad (2.6)$$

PROOF. Without loss of generality we may assume  $a = 0$ . The surjectivity of  $A$  implies that the restriction  $A|_{N_2}$  of  $A$  to the closed subspace  $N_2$  is bijective; by the Banach open mapping theorem its inverse is also continuous and equals the rightinverse  $A^+$  of  $A$ :  $AA^+y = y$  for each  $y \in Y$ . The rightinverse of  $A$  equals  $A(AA^*)^{-1}$  [4]. Let  $\varphi := f|_{N_2}$ . This mapping satisfies the conditions of the Kantorovich theorem:

- (i)  $D\varphi(0) = A|_{N_2} \in L(N_2, Y)$  is invertible and  $\|D\varphi(0)^{-1}\| = \|A^+\| \leq \lambda^{-1}$ ;
- (ii)  $\|D\varphi(x) - D\varphi(y)\| \leq \|Df(x) - Df(y)\| \leq L\|x - y\|$ ,  $x, y \in N_2 \cap B(0, r)$ ;
- (iii)  $\|D\varphi(0)^{-1} \varphi(0)\| = \tilde{\gamma} \leq \gamma$ ;
- (iv)  $\kappa = L\gamma\lambda^{-1} < \frac{1}{2}$  and  $\rho = \lambda(1 - \sqrt{1 - 2\kappa})/L < r$ .

Hence the equation  $\varphi(x) = 0$  has a solution  $z$  in  $B(0, \rho) \cap N_2$  and this solution is unique in  $B(0, \rho_2) \cap N_2$  with  $\rho_2$  as given in (2.4) [8]. This  $z$ , zero of  $f$  in  $B(0, \rho)$  is used to obtain a lowerbound  $\rho_1$  of  $d(0, S)$ . Since  $Df$  is Lipschitz continuous on  $B(0, r)$  we have [17]

$$\|f(z) - f(0) - Az\| \leq \frac{1}{2} L\|z\|^2$$

and consequently

$$\tilde{\gamma} = \|A^+f(0)\| = \|z + A^+(f(z) - f(0) - Az)\| \leq \|z\| + \frac{1}{2} L\lambda^{-1}\|z\|^2.$$

Hence the positive zero  $\rho_3$  of the quadratic function

$$t \mapsto \frac{1}{2} L\lambda^{-1} t^2 + t - \tilde{\gamma}$$

is majorized by  $\|z\|$ . With (iv) we find  $\|z\| < \rho < 2\gamma$  and consequently

$$L\|z\| < 2L\gamma < \lambda.$$

Let be  $V := \{x \in X \mid \|x\| < \|z\|\}$  and

$$P(x) := Df(x)|_{N_1}, \quad Q(x) := Df(x)|_{N_2}, \quad x \in V.$$

If  $y = y_1 + y_2$ ,  $y_i \in N_i$  ( $i = 1, 2$ ) then

$$Df(x)y = Df(x)(y_1 + y_2) = P(x)y_1 + Q(x)y_2.$$

Since  $P(0) = 0$  we derive from the Lipschitz continuity

$$\|P(x)\| = \|P(x) - P(0)\| \leq \|Df(x) - Df(0)\| \leq L\|x\| \leq L\|z\|, \quad x \in V$$

and similarly

$$\|Q(x) - Q(0)\| \leq \|Df(x) - Df(0)\| \leq L\|z\|, \quad x \in V.$$

But  $Q(0) = D\varphi(0)$  and so

$$\|(Q(x) - Q(0))Q(0)^{-1}\| \leq L\|z\| \lambda < 1, \quad x \in V,$$

which implies that  $Q(x) = (I + (Q(x) - Q(0))Q(0)^{-1})Q(0)$  is invertible for each  $x \in V$ . With Banach's theorem [13] we obtain readily an upperbound, say  $\alpha$ , of  $\|Q(x)^{-1}P(x)\|$ :

$$\|Q(x)^{-1}P(x)\| \leq \|Q(0)^{-1}\| \|Px\| / (1 - L\|z\|\lambda^{-1}) \leq \frac{L\|z\|}{\lambda - L\|z\|} =: \alpha, \quad x \in V \quad (2.7)$$

Let be  $\Omega := \{x \in V \mid f(x) = 0\}$ . From the uniqueness of the zero  $z$  of  $\varphi$  in  $B(0, \rho) \cap N_2$  we conclude that  $\|x\| \leq \|z\| \wedge x \in N_2 \wedge f(x) = 0$  implies  $x = z$ .

Now we assume  $w = w_1 + w_2 \in \Omega$ , with  $w_i \in N_i$  ( $i = 1, 2$ ) and  $w_1 \neq 0$ .

In our investigations we use the line  $\ell$  in  $N_1$ :

$$\ell := \{(1 - t\|w_1\|^{-1})w_1 \mid t \in \mathbb{R}\}$$

and the function  $F$  defined as follows:

$$\mathbb{R} \times N_2 \ni (t, y_2) \mapsto F(t, y_2) := f((1 - t\|w_1\|^{-1})w_1 + y_2), t^2 + \|y_2\|^2 < r^2.$$

It is clear that  $F(0, w_2) = 0$ . Since

$$F(0, w_2 + h_2) - F(0, w_2) - Df(w)h_2 = o(\|h_2\|), \quad (h_2 \rightarrow 0),$$

the derivative  $D_2 F(0, w_2)$  of  $F$  in  $(0, w_2)$  with respect to  $w_2$  equals  $Df(w)|_{N_2} = Q(w)$  and this derivative is a regular element of  $\mathcal{L}(N_2, Y)$ . In conformity with the implicit function theorem [20], there exists an interval  $I \subset \mathbb{R}$  around 0 and a differentiable mapping  $\psi: I \rightarrow N_2$  such that  $\psi(0) = w_2$ ,  $F(t, \psi(t)) = 0$  and too for each  $t \in I$

$$D\psi(t) = -(D_2 F(t, \psi(t)))^{-1} D_1 F(t, \psi(t)).$$

( $D_1 F(t, \psi(t))$  being the derivative of  $F$  in  $(t, \psi(t))$  with respect to  $t$ ). For  $\delta$  small enough the differentiable curve

$$\{(1 - t\|w_1\|^{-1})w_1 + \psi(t) \in X \mid t \in (-\delta, \delta) \subset I\}$$

is a subset of  $V$ . On this interval  $(-\delta, \delta)$  we have

$$\psi(t) - \psi(0) = - \int_0^t D_2 F(\tau, \psi(\tau))^{-1} D_1 F(\tau, \psi(\tau)) d\tau. \quad (2.8)$$

By the definition of  $F$

$$D_1 F(t, y_2) = -P((1 - t\|w_1\|^{-1})w_1 + y_2)\|w_1\|^{-1} w_1$$

so

$$\|D_1 F(t, y_2)\| \leq \|P((1 - t\|w_1\|^{-1})w_1 + y_2)\|.$$

Hence for  $|t| < \delta$  we obtain with (2.7)

$$\|D_2 F(t, \psi(t))^{-1} D_1 F(t, \psi(t))\| \leq \|Q((1 - t\|w_1\|^{-1})w_1 + \psi(t))^{-1} P((1 - t\|w_1\|^{-1})w_1 + \psi(t))\| \leq \alpha \quad (2.9)$$

With this upperbound we find easily from (2.8)

$$\|\psi(t) - w_2\| \leq \alpha|t|, \quad |t| < \delta. \quad (2.10)$$

Let us suppose that (, compare with (2.3),)

$$w \in W := \{x = x_1 + x_2 \in V \mid \alpha\|x_1\| + \|x_2\| < \|z\|, x_i \in N_i, i = 1, 2\} \quad (2.11)$$

Then, as follows from lemma 2.1, for each  $t \in [0, \|w_1\|]$

$$(1 - t\|w_1\|^{-1})w_1 + x_2 \in B(0, \|z\|) \quad (2.12)$$

if  $x_2 \in B(w_2, \alpha t)$ . So if  $w \in W$  than the function  $\psi$  (, solving  $y_2$  from the equation  $F(t, y_2) = 0$  as a function of  $t$ ,) can be continued to the right until  $t = \|w_1\|$ . Analogously to (2.10) one finds for this extended mapping  $\psi$ :

$$\|\psi(\|w_1\|) - w_2\| \leq \alpha\|w_1\|.$$

But

$$0 = F(\|w_1\|, \psi(\|w_1\|)) = f(\psi(\|w_1\|)) = \varphi(\psi(\|w_1\|)),$$

for  $\psi(\|w_1\|) \in N_2$ . With (2.12) we obtain

$$\psi(\|w_1\|) \in B(0, \|z\|).$$

This conclusion contradicts that  $z$  is the unique zero of  $\psi$  in the ball  $B(0, \rho_2) \cap N_2$ . The contradiction proves the incorrectness of assumption (2.11)! We have proved that

$$\Omega \subset W^c = \{x = x_1 + x_2 \in X \mid \alpha\|x_1\| + \|x_2\| \geq \|z\|, x_i \in N_i, i = 1, 2\} \quad (2.13)$$

The distance  $d(0, \Omega)$  is minorized by  $d(0, W^C)$ ; with lemma 2.2 one finds

$$d(0, \Omega) \geq \frac{\|z\|}{\sqrt{1+\alpha^2}} = \frac{\|z\|(\lambda - L\|z\|)}{\sqrt{L^2\|z\|^2 + (\lambda - L\|z\|)^2}} = \|z\| \left(1 + \left(\frac{v}{1-v}\right)^2\right)^{-\frac{1}{2}},$$

where  $v = L\|z\|/\lambda \in [L\rho_3\lambda^{-1}, L\rho\lambda^{-1}) \subset (0, 1)$ . Since the function

$$v \rightarrow \left(1 + \left(\frac{v}{1-v}\right)^2\right)^{-\frac{1}{2}}$$

on this interval attains its minimum in the endpoint  $L\rho_3\lambda^{-1}$ , we finally obtain

$$d(0, \Omega) \geq \rho_3(1 + L^2\rho_3^2(\lambda - L\rho_3)^{-2})^{-\frac{1}{2}} = \rho_1. \quad \square$$

REMARK 1. Under the same conditions as in theorem 2.1, if  $\kappa < \frac{1}{2}$ , then the iterates

$$x_{k+1} := x_k - A^*(AA^*)^{-1} f(x_k), \quad k \in \mathbb{N}, x_1 := 0, \quad (2.14)$$

are defined for all  $k$ , and converge to  $z \in B(0, \rho) \cap N_2$  [20].

This modified Newton's method gives a linearly convergent sequence  $\{x_k\} \subset B(0, \rho_2) \cap N_2$  and

$$\|x_k - z\| \leq 2 \frac{\lambda}{L} (1 - \sqrt{1-2\kappa})^{k+1}. \quad \square$$

REMARK 2. The bounds for the zero of  $f|_{N_2}$  can be obtained by a direct analysis of the mapping  $\varphi = f|_{N_2}$ . If  $x \in N_2$ , then there exists a unique  $y \in Y$  such that  $x = A^*y$ ; hence if  $\hat{y} \in Y$  is a zero of  $K: Y \rightarrow Y$ , where  $K(y) = f(A^*y)$ , then  $z = A^*\hat{y} \in N_2$  is a zero of  $f$ . That  $\hat{y}$  can be obtained with modified Newton's method:

$$y_{k+1} := y_k - (AA^*)^{-1} f(A^*y_k), \quad k \in \mathbb{N}, y_1 := 0 \quad (2.15)$$

Evidently, the sequences  $\{x_k\}$  and  $\{y_k\}$ , given in (2.14) and (2.15) are



strongly related:  $x_k = A^* y_k$ .

□

REMARK 3. Quite the same results can be derive for  $f: X \rightarrow Y$  where  $X$  and  $Y$  are Hilbert space over the field of complex numbers. This will occur in section 4 where the results of this section are applied to the eigenproblem. □

### 3. DISTRIBUTION OF ERRORS IN A LINEAR CONTINUOUS SYSTEM

#### 3.1.

Let be given a real continuous function  $x \in C(I)$ ,  $I$  being the interval  $[0,1]$  and a real differentiable function  $y \in C^1(I)$ . Further we assume to have at our disposal the numbers  $a, b \in \mathbb{R}$ . This quadruple  $(a, b, x, y)$  defines residual  $r \in C(I)$ :

$$r(t) := \dot{y}(t) - ay(t) - bx(t), \quad 0 \leq t \leq 1. \quad (3.1)$$

In order to evaluate how far the pair  $(x, y)$  fits in the scalar linear model

$$\dot{y} = ay + bx \quad (3.2)$$

we introduce perturbations

$$\varphi = (\alpha, \beta, \rho, \xi, \eta) \in X := C(I)^4 \times C^1(I) \quad (3.3)$$

such that

$$\dot{y}(t) + \dot{\eta}(t) = (a + \alpha(t))(y(t) + \eta(t)) + (b + \beta(t))(x(t) + \xi(t)) + \rho(t) \quad (3.4)$$

So this quintuple  $\varphi$  has to satisfy the equation  $f(\varphi) = 0$  where  $f: X \rightarrow Y$ ,  $Y := C(I)$ , with

$$\begin{aligned} f(\varphi)(t) &= f(\alpha, \beta, \rho, \xi, \eta)(t) = \dot{y}(t) + \dot{\eta}(t) - (a + \alpha(t))(y(t) + \eta(t)) - \\ &\quad (b + \beta(t))(x(t) + \xi(t)) - \rho(t) \\ &= r(t) + \dot{\eta}(t) - a\eta(t) - b\xi(t) - \alpha(t)y(t) - \beta(t)x(t) - \\ &\quad \rho(t) - \alpha(t)\eta(t) - \beta(t)\xi(t). \end{aligned} \quad (3.5)$$

The modification of (3.2) into

$$\dot{\eta}(t) = (a+\alpha(t))\eta(t) + \beta(t)(x(t)+\xi(t)) + b\xi(t) + \alpha y(t) + \rho(t) - r(t) \quad (3.6)$$

gives a weakly nonlinear system [6].

The functions  $\alpha$  and  $\beta$  are considered to be fluctuations of the structural parameters  $a$  and  $b$  respectively. The functions  $\xi$  and  $\eta$  are perturbations of the given functions  $x$  and  $y$  and  $\rho$  is a corrected residual. In this view the description of the error in the model with residual  $r$ , has been replaced by a description with the fluctuations  $\alpha$  and  $\beta$ , the perturbations  $\xi$  and  $\eta$  and the corrected residual  $\rho$ . In a natural way the question rises to determine the minimal  $\varphi = (\alpha, \beta, \rho, \xi, \eta)$ , with respect to some norm, such that  $a$ ,  $b$ ,  $x$  and  $y$  fit in system (3.4).

The same problem can be seen as an example of optimal control. The determination of an minimal  $\varphi = (\alpha, \beta, \rho, \xi, \eta)$  (, with respect to some norm) subject to condition (3.6) is just a weakly nonlinear tracking problem. In the linear state equation (3.6) the state  $\eta$  and the controls  $\alpha$ ,  $\beta$ ,  $\xi$  and  $\rho$  occur in a bilinear way [1,6,11,12,21].

### 3.2.

In this section we apply the approach of the preceding section for the estimation of that minimal  $\varphi$ .

As we see from (3.6)

$$f(\varphi_1 + \varphi_2)(t) = f(\varphi_1)(t) + Df(\varphi_1) \varphi_2(t) + R(\varphi_1; \varphi_2)(t) \quad (3.7)$$

where

$$Df(\varphi_1) \varphi_2(t) = \dot{\eta}_2 - (a + \alpha_1)\eta_2 - (b + \xi_1)\xi_2 - \alpha_2(y + \eta_1) - \beta_2(x + \xi_1) - \rho_2 \quad (3.8)$$

and

$$R(\varphi_1; \varphi_2)(t) = -\alpha_2\eta_2 - \beta_2\eta_2 \quad (3.9)$$

with  $\varphi_i = (\alpha_i, \beta_i, \rho_i, \xi_i, \eta_i)$ ,  $i = 1, 2$ .

Both  $\varphi_1$  and increment  $\varphi_2$  are elements of the productspace  $X$  with inner-product

$$\langle \varphi_1, \varphi_2 \rangle := (\alpha_1, \alpha_2) + (\beta_1, \beta_2) + (\rho_1, \rho_2) + (\xi_1, \xi_2) + (\eta_1, \eta_2),$$

$$\varphi_1, \varphi_2 \in X,$$

where  $\varphi_i = (\alpha_i, \beta_i, \rho_i, \xi_i, \eta_i)$ ,  $i = 1, 2$ .

With the  $L_2$ -norms in the prehilbert spaces  $X$  and  $Y$  we find for  $f: X \rightarrow Y$  that

$$R(\varphi_1, \varphi_2) \|\varphi_2\|^{-1} \rightarrow 0 \quad (\varphi_2 \rightarrow 0),$$

for each  $\varphi_1 \in X$ . Nevertheless  $f$  is not Frechet differentiable at  $\varphi_1 \in X$ , for the linear mapping  $Df(\varphi_1): X \rightarrow Y$  is not bounded; on the other hand  $f$  is Gateaux differentiable at each  $\varphi \in X$ .

The constant term  $r$  of  $f(\varphi)$  equals  $f(0)$  and the linear part, relatively the  $L_2$ -norms in  $X$  and  $Y$ , is  $A\varphi(t) := Df(0)\varphi(t) = \dot{\eta}(t) - a\eta(t) - b\xi(t) - \alpha(t)y(t) - \beta(t)x(t) - \rho(t)$ .

The domain  $X$  of the Gateaux differential  $A$  of  $f$  at  $0$  is dense in  $H := L_2(I)$  and its range  $Y$  is dense in  $L_2(I)$ .

Indeed the mapping  $f$  is surjective as is  $A$ : for each  $p \in Y$  we have  $f(0, 0, r-p, 0, 0) = A(0, 0, -p, 0, 0) = p$ .

Now we start to construct the adjoint  $A^*$  of  $A$ . Suppose  $\psi \in D(A^*)$  and let be  $A^*\psi = (\alpha_1, \beta_1, \rho_1, \xi_1, \eta_1)$ . This means that for each  $(\alpha, \beta, \rho, \xi, \eta) \in X$  holds

$$\begin{aligned} (A(\alpha, \beta, \rho, \xi, \eta), \psi) &= \int_0^1 (\dot{\eta} - a\eta - b\xi - \alpha y - \beta x - \rho)(t) \psi(t) dt \\ &= ((\alpha, \beta, \rho, \xi, \eta), (\alpha_1, \beta_1, \rho_1, \xi_1, \eta_1)) = \int_0^1 (\alpha\alpha_1 + \beta\beta_1 + \rho\rho_1 + \xi\xi_1 + \eta\eta_1)(t) dt \end{aligned}$$

Thus for each  $(\alpha, \beta, \rho, \xi, \eta) \in X$  we have

$$\int_0^1 [(\dot{\eta} - a\eta)\psi - \eta\eta_1](t) dt = \int_0^1 [(b\psi + \xi_1)\xi + (y\psi + \alpha_1)\alpha + (x\psi + \beta_1)\beta + (\psi + \rho_1)\rho](t) dt \quad (3.10)$$

If in the quintuple  $(\alpha, \beta, \rho, \xi, \eta) \in X$  each element is taken zero except respectively  $\alpha$ ,  $\beta$ ,  $\rho$  and  $\xi$  we find

$$\begin{aligned} \alpha_1(t) &= -y(t)\psi(t), \beta_1(t) = -x(t)\psi(t), \rho_1(t) = -\psi(t), \xi_1(t) = -b\psi(t), \\ 0 &\leq t \leq 1 \end{aligned} \quad (3.11)$$

If  $\alpha = \beta = \rho = \xi = 0$  we find with (3.10) that for each  $\eta \in C^1(I)$ :

$$\int_0^1 [(\dot{\eta} - a\eta)\psi - \eta\eta_1] dt = 0.$$

Since the adjoint of  $\eta \mapsto \dot{\eta}$  is the mapping  $\psi \mapsto -\dot{\psi}$  with  $\psi(0) = \psi(1) = 0$  [10], and that of  $\eta \mapsto a\eta$  is  $\psi \mapsto a\psi$  we find

$$\eta_1(t) = -\dot{\psi}(t) - a\psi(t), \quad \psi(0) = \psi(1) = 0.$$

So

$$\left\{ \begin{aligned} D(A^*) &= \{\psi \in C^1(I) \mid \psi(0) = \psi(1) = 0\} \\ A^*\psi(t) &= (-y(t)\psi(t), -x(t)\psi(t), -\psi(t), -b\psi(t), -\dot{\psi}(t) - a\psi(t)), \quad 0 \leq t \leq 1. \end{aligned} \right.$$

Thus far the mappings  $f$  and  $Df(\psi)$  have been considered on the domain  $X$ ; the nonlinear term  $\beta\xi$  in the expression (3.5) of  $f(\varphi)$  prevents  $f$  to be defined on the Hilbert space  $L_2(1)^5 \times H^1(I)$ . Consistently, we now consider the restriction of  $f$  to

$$N_2 := R(A^*) = \{(-y\psi, -x\psi, -\psi, -b\psi, -\dot{\psi} - a\psi) \mid \psi \in C^1(I), \psi(0) = \psi(1) = 0\}.$$

A usable upperbound of  $\inf\{\|\varphi\|_2 \mid \varphi \in X, f(\varphi) = 0\}$  can be derived from a zero of  $f|_{N_2}$ .

Let  $\varphi \in N_2$ . Then there exists a  $\psi \in C^1(I)$ , with  $\psi(0) = \psi(1) = 0$ , such that

$$\varphi(t) = (-y(t)\psi(t), -x(t)\psi(t), -\psi(t), -b\psi(t), -\dot{\psi}(t) - a\psi(t)), \quad t \in I. \quad (3.12)$$

Hence  $f(\emptyset) = 0$  implies

$$\begin{cases} K(\psi)(t) := r(t) - \bar{\psi}(t) + g(t)\psi(t) - y(t)\psi(t)\bar{\psi}(t) - k(t)\psi^2(t) = 0 \end{cases} \quad (3.13)$$

$$\begin{cases} \psi(0) = \psi(1) = 0 \end{cases} \quad (3.14)$$

where

$$g(t) := a^2 + b^2 + 1 + x^2(t) + y^2(t), \quad k(t) := ay(t) + bx(t), \quad t \in I.$$

The solution of the equation  $f(\psi) = 0$  in  $N_2$  can be derived with (3.12) from the solution of the two point boundary value problem (TPBVP) (3.13), (3.14). So we investigate the existence problem for this TPBVP. Instead of the Newton approach we make use of the Banach fixed point theorem in order to proof the existence and to estimate the solution of this TPBVP [3], [5].

3.3.

For the proof of the existence of a solution  $\psi \in C^1([0,1])$  of TPBVP (3.13), (3.14) we consider the differential operator  $P: V \rightarrow C([0,1])$ , where

$$\begin{cases} V := \{\psi \in C^2[0,1] \mid \psi(0) = \psi(1) = 0\} \end{cases} \quad (3.15)$$

$$\begin{cases} P\psi = w \Leftrightarrow \bar{\psi}(t) - g(t)\psi(t) = w(t), \quad t \in [0,1] \end{cases} \quad (3.16)$$

Since  $g(t) > 0$ ,  $0 \leq t \leq 1$ , the linear operator  $P$  is regular [7].

With respect to the supremum norm  $\|\cdot\|_\infty$  on  $C[0,1]$  operator  $P$  has a bounded inverse  $P^{-1}: C[0,1] \rightarrow V$  such that [10]

$$\|P^{-1}\|_\infty \leq \frac{1}{m(g)}, \quad (3.17)$$

where

$$m(g) := \inf\{g(t) \mid t \in [0,1]\}. \quad (3.18)$$

Thus for the solution  $u$  of TPBVP



$$\bar{\psi} - g\psi = w, \quad \psi(0) = \psi(1) = 0$$

we have

$$\|\psi\|_{\infty} \leq \frac{\|w\|_{\infty}}{m(g)}$$

Further holds [7]

$$\psi(t) = \int_0^1 G(t,s) (g(s)\psi(s) + w(s)) ds, \quad (3.19)$$

where

$$G(t,s) = \begin{cases} (t-1)s, & 0 \leq s \leq t \leq 1 \\ t(s-1) & 0 \leq t \leq s \leq 1 \end{cases} \quad (3.20)$$

Now we investigate the mapping  $T: C^1[0,1] \rightarrow C^1[0,1]$ , where

$$Tv = u \Leftrightarrow \begin{cases} u - gu = r - yv\dot{v} - kv^2 \\ u(0) = u(1) = 0. \end{cases} \quad (3.21)$$

Hence

$$u = Tv = P^{-1}(r - yv\dot{v} - kv^2) \quad (3.22)$$

and

$$u(t) = \int_0^1 G(t,s) (gu + r - yv\dot{v} - kv^2)(s) ds \quad (3.23)$$

Consequently any point  $\psi \in C^1[0,1]$  is a solution of TPBVP (3.13), (3.14) if and only if  $\psi = T\psi$ .

With respect to the norm

$$\|v\|_S := \|v\|_{\infty} + \|\dot{v}\|_{\infty}, \quad v \in C^1[0,1] \quad (3.24)$$

the linear space  $C^1[0,1]$  is complete.

In  $C^1[0,1], \|\cdot\|_s$  we determine a closed ball  $W := \overline{B(0,p)}$  such that  $T(W) \subset W$ . Equation (3.22) implies

$$\|u\|_\infty \leq (\|r\|_\infty + \|y\|_\infty \|v\|_\infty \|\dot{v}\|_\infty + \|k\|_\infty \|v\|_\infty^2) / m(g). \quad (3.25)$$

From (3.19) we derive

$$\dot{u}(t) = \int_0^1 \frac{\partial G}{\partial t}(t,s) (gu + r - yv\dot{v} - kv^2)(s) ds,$$

where

$$\frac{\partial G}{\partial t}(t,s) = \begin{cases} s, & 0 \leq s \leq t \leq 1 \\ s-1, & 0 \leq t \leq s \leq 1. \end{cases}$$

Consequently

$$\|\dot{u}\|_\infty \leq \frac{3}{2} [\|g\|_\infty \|u\|_\infty + \|r\|_\infty + \|y\|_\infty \|v\|_\infty \|\dot{v}\|_\infty + \|k\|_\infty \|v\|_\infty^2]. \quad (3.26)$$

Addition of (3.25) and (3.26) gives

$$\begin{aligned} \|u\|_s &\leq c(g) [\|r\|_\infty + \|y\|_\infty \|v\|_\infty \|\dot{v}\|_\infty + \|k\|_\infty \|v\|_\infty^2] \\ &\leq c(g) [\|r\|_\infty + (\|y\|_\infty + \|k\|_\infty) \|v\|_s^2] \end{aligned} \quad (3.27)$$

where

$$c(g) := \frac{3}{2} \left( \frac{\|g\|_\infty}{m(g)} + 1 \right) + \frac{1}{m(g)}. \quad (3.28)$$

Thus  $\|v\|_s \leq p$  implies  $\|Tv\|_s \leq p$  if

$$c(g) (\|r\|_\infty + (\|y\|_\infty + \|k\|_\infty) p^2) < p.$$

The last inequality is satisfied if and only if

$$p_1 \leq p \leq p_2 \quad (3.29)$$

where

$$p_i = \frac{1 + (-1)^i \sqrt{1 - 4c^2(g)(\|y\|_\infty + \|k\|_\infty)\|x\|_\infty}}{2c(g)(\|y\|_\infty + \|k\|_\infty)}, \quad i = 1, 2. \quad (3.30)$$

Evidently, residual  $r$  has to be sufficient small:

$$\|r\|_\infty \leq (4c^2(g)(\|y\|_\infty + \|k\|_\infty))^{-1} \quad (3.31)$$

Now we investigate  $T:W \rightarrow W$  ( $p_1 \leq p \leq p_2$ ) on its contraction property.

Let be  $u_i = Tv_i$ ,  $v_i \in W$ ,  $i = 1, 2$ .

Then

$$u_1 - u_2 = p^{-1}(y(v_2 \dot{v}_2 - v_1 \dot{v}_1) + k(v_2^2 - v_1^2))$$

and

$$\dot{u}_1(t) - \dot{u}_2(t) = \int_0^1 \frac{\partial G}{\partial t}(t, s)(g(u_1 - u_2) + y(v_2 \dot{v}_2 - v_1 \dot{v}_1) + k(v_2^2 - v_1^2))(s) ds.$$

Since

$$\begin{cases} v_2 \dot{v}_2 - v_1 \dot{v}_1 = v_2(\dot{v}_2 - \dot{v}_1) + (v_2 - v_1)\dot{v}_1 \\ v_2^2 - v_1^2 = (v_2 + v_1)(v_2 - v_1), \end{cases}$$

evidently

$$\begin{cases} \|v_2 \dot{v}_2 - v_1 \dot{v}_1\|_\infty \leq (\|v_1\|_s + \|v_2\|_s)\|v_1 - v_2\|_s \\ \|v_2^2 - v_1^2\|_\infty \leq (\|v_1\|_s + \|v_2\|_s)\|v_1 - v_2\|_s. \end{cases}$$

Then

$$\|Tv_1 - Tv_2\|_\infty = \|u_1 - u_2\|_\infty \leq \frac{1}{m(g)} (\|y\|_\infty + \|k\|_\infty) (\|v_1\|_s + \|v_2\|_s) \|v_1 - v_2\|_s \quad (3.32)$$

and

$$\begin{aligned} |\dot{u}_1(t) - \dot{u}_2(t)| &\leq \frac{3}{2} (\|g\|_\infty \|u_1 - u_2\|_\infty + \|y\|_\infty \|v_2 - v_1\|_\infty + \|k\|_\infty \|v_2^2 - v_1^2\|_\infty) \\ &\leq \frac{3}{2} \left( \frac{\|g\|_\infty}{m(g)} + 1 \right) (\|y\|_\infty + \|k\|_\infty) (\|v_1\|_s + \|v_2\|_s) \|v_1 - v_2\|_s. \end{aligned} \quad (3.33)$$

Addition of (3.32) and (3.33) gives,

$$\begin{aligned} \|Tv_1 - Tv_2\|_s &\leq c(g) (\|y\|_\infty + \|k\|_\infty) (\|v_1\|_s + \|v_2\|_s) \|v_1 - v_2\|_s \\ &\leq 2pc(g) (\|y\|_\infty + \|k\|_\infty) \|v_1 - v_2\|_s. \end{aligned}$$

So the mapping  $T:W \rightarrow W$  is a contraction if

$$p < p_3 := \frac{1}{2c(g) (\|y\|_\infty + \|k\|_\infty)} \quad (3.33)$$

Remark that, provided condition (3.31) is satisfied,  $p_1 \leq p_3 \leq p_2$ . Now we have reached the following result.

The mapping  $T:W = \overline{B(0,p)} \rightarrow W$  is a contraction for each  $p \in [p_1, p_3]$  if

$$\|r\|_\infty \leq \frac{1}{4c^2(g) (\|y\|_\infty + \|k\|_\infty)}.$$

Under these conditions  $T$  has a unique fixed point  $\psi$  in  $\overline{B(0,p)}$ . Moreover, the sequence  $\{u_n\}$ , with  $u_0 = 0$  and

$$u_n = Tu_{n-1}, \quad n \in \mathbb{N},$$

converges in  $C^1[0,1], \|\cdot\|_s$  to that solution  $\psi$  of the equation  $u = Tu$ . For that solution  $\psi$  holds the inequality

$$\|\psi\|_s \leq \frac{\|u_1\|_s}{1 - 2pc(g) (\|y\|_\infty + \|k\|_\infty)}$$

The first iterate  $u_1$  is solution of the TPBVP

$$\ddot{u}_1 - gu_1 = r, \quad u_1(0) = u_1(1) = 0.$$

As a consequence of (3.27)

$$\|u_1\|_{\infty} \leq c(g) \|r\|_{\infty}.$$

Therefore

$$\|\psi\|_s \leq \frac{c(g)}{1-2p_1 c(g) (\|y\|_{\infty} + \|k\|_{\infty})} \|r\|_{\infty}. \quad (3.34)$$

With  $p = p_1$  finally we have

$$\|\psi\|_s \leq \frac{c(g)}{1-2p_1 c(g) (\|y\|_{\infty} + \|k\|_{\infty})} \|r\|_{\infty}.$$

For the solution  $\phi := (-y\psi, -x\psi, -\psi, -b\psi, -\dot{\psi}-a\psi)$  of equation  $f(\phi) = 0$  we have

$$\begin{aligned} \int_0^1 \phi^2(t) dt &= \int_0^1 [(y^2 + x^2 + 1 + b^2 + a^2)\psi^2 + \dot{\psi}^2 + 2a\psi\dot{\psi}](t) dt \\ &= \int_0^1 (g(t)\psi^2(t) + \dot{\psi}^2(t)) dt \end{aligned}$$

since  $\psi(0) = \psi(1) = 0$ . Now

$$\|\phi\|_2^2 \leq \|g\|_{\infty} (\|\psi\| + \|\dot{\psi}\|)^2 = \|g\|_{\infty} \|\psi\|_s^2$$

Consequently

$$\|\phi\|_s \leq \frac{c(g) \sqrt{\|g\|_{\infty}}}{1-2p_1 c(g) (\|y\|_{\infty} + \|k\|_{\infty})} \|r\|_{\infty}.$$

This result is summarized in

**THEOREM 3.1.** Let be given  $a, b \in \mathbb{R}$ ,  $x \in C(I)$  and  $y \in C^1(I)$ , where  $I = [0, 1]$ .  
Let

$$r(t) := \dot{y}(t) - ay(t) - bx(t), \quad g(t) := a^2 + b^2 + 1 + x^2(t) + y^2(t), \quad k(t) =$$

$$ay(t) + bx(t), \quad t \in I.$$

If

$$4\|r\|_{\infty} c^2(g) (\|y\|_{\infty} + \|k\|) < 1,$$

where

$$c(g) = \frac{3}{2} \left( \frac{\|g\|_{\infty}}{m(g)} + 1 \right) + \frac{1}{m(g)}$$

with

$$m(g) = \inf\{g(t) \mid t \in I\},$$

then there exists a perturbation  $\phi = (\alpha, \beta, \rho, \xi, \eta) \in C(I)^4 \times C^1(I)$  such that

$$\dot{y} + \dot{\eta} = (a+\alpha)(y+\eta) + (b+\beta)(x+\xi) + \rho$$

and

$$\|\phi\|_2 \leq \frac{c(g) \sqrt{\|g\|_{\infty}}}{1 - 2p_1 c(g) (\|y\|_{\infty} + \|k\|_{\infty})} \|r\|_{\infty}$$

where

$$p_1 = \frac{1 - \sqrt{1 - 4c^2(g) (\|y\|_{\infty} + \|k\|_{\infty}) \|r\|_{\infty}}}{2c(g) (\|y\|_{\infty} + \|k\|_{\infty})}.$$

□

REMARK. In a next memorandum we will analyse the vectorial analogue of the distribution problem and also will be described a numerical method for the computation of these suboptimal fluctuations and perturbations.



#### 4. A POSTERIORI ERROR BOUNDS FOR AN EIGENPAIR AND NUMERICAL RESULTS.

In this section we apply the results of section two to the eigenproblem. We assume to have at our disposal an approximate eigenproblem solution  $(\lambda, x)$  for a given complex  $n \times n$  matrix. For instance, this eigenpair can be seen as the result of some eigenvalue computations performed with finite arithmetic.

Let be  $r := Ax - \lambda x$ , the residual vector corresponding to the approximate eigenpair  $(\lambda, x)$ ; further we assume  $x$  to be normalized in length:  $x^* x = 1$ . The pair  $(\lambda, x)$  has been afflicted with errors  $(\lambda \neq 0)$  so at the moment we introduce perturbations  $(\delta, h)$  in order to compensate these errors, i.e.  $(\lambda + \delta, x + h)$  has to be an exact eigenpair for matrix  $A$ .

For the analysis of the bounds for the necessary corrections  $(\delta, h)$  we introduce the nonlinear, smooth function  $f : C \times C^n \rightarrow C^n$ , defined by

$$f(\delta, h) := A(x+h) - (\lambda + \delta)(x+h). \quad (4.1)$$

Then,  $f(0,0) = r$  and for  $P := Df(0,0)$  we find

$$P(\delta, h) = (A - \lambda I)h - \delta x. \quad (4.2)$$

With respect to the usual innerproducts in  $C^n$  and  $C^{n+1} \sim C \times C^n$  we derive for the adjoint  $P^* : C^n \rightarrow C \times C^n$  of  $Q$ :

$$\begin{aligned} (P^* k, (\delta, h)) &= (k, P(\delta, h)) = (k, (A - \lambda I)h - \delta x) \\ &= (k, -x)\bar{\delta} + ((A - \lambda I)^* k, h), \quad \delta \in C; h, k \in C^n. \end{aligned}$$

So  $P^* k = ((k, -x), (A - \lambda I)^* k) \in C \times C^n$  and with (4.2) we obtain

$$PP^* k = (A - \lambda I)(A - \lambda I)^* k - (k, -x)x = (A - \lambda I)(A - \lambda I)^* k + xx^* k.$$

For the complex  $n \times n$  matrix  $QQ^*$  we have found

$$PP^* = (A - \lambda I)(A - \lambda I)^* + xx^*. \quad (4.3)$$

If  $\lambda$  is not an eigenvalue of  $A$ , then  $PP^*$  is regular. If  $(\lambda, x)$  is an eigen-

pair, then the algebraic multiplicity of  $\lambda$  is decisive for the regularity of  $PP^*$ .

LEMMA 4.1. Let be  $(\lambda, x)$  an eigenpair of matrix  $A$ . Then  $PP^* = (A - \lambda I)(A - \lambda I)^* + xx^*$  is regular if and only if the algebraic multiplicity of  $\lambda$  is equal to one.

PROOF. Without loss of generality we may assume  $A - \lambda I$  to be an uppertriangular matrix and  $x$  to be  $e_1$ , the first unit vector. Moreover, as concerns the eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n$  of  $A$  we assume  $|\lambda_2 - \lambda| \leq |\lambda_3 - \lambda| \leq \dots \leq |\lambda_n - \lambda|$ . Let

$$P = (A - \lambda I : x) = \begin{pmatrix} 0 & a_{12} & a_{13} & \dots & a_{1n} & 1 \\ 0 & \lambda_2 - \lambda & a_{23} & \dots & a_{2n} & 0 \\ 0 & 0 & \lambda_3 - \lambda & \dots & a_{3n} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_n - \lambda & 0 \end{pmatrix}.$$

Evidently,  $\text{rank } P = n$  if and only if  $\lambda_2 \neq \lambda$ . □

After these preparations an obvious application of theorem 2.1 is formulated in

THEOREM 4.1. Let  $f, P : C \times C^n \rightarrow C^n$  be defined by (4.1) and (4.2). If

- (i)  $\text{rank } P = n$ ,  $\|(PP^*)^{-1}\|_2^{1/2} = \mu_n^{-1} \leq \mu^{-1}$ ,  $\mu_n$  being the smallest singular value of  $P$ ;
- (ii)  $(r^*(PP^*)^{-1}r)^{1/2} = \tilde{\gamma} < \gamma$ , where  $r = f(0,0) = Ax - \lambda x$ ;
- (iii)  $\kappa = \gamma\mu^{-1} < \frac{1}{2}$ ,

then the distance  $d$  of  $(0,0)$  to  $S = \{(\delta, h) \in C \times C^n | f(\delta, h) = 0\}$  satisfies the inequalities

$$\rho_1 := \rho_3 \left(1 + \frac{\rho_3^2}{(\mu - \rho_3)^2}\right)^{-\frac{1}{2}} < d < \rho, \quad \text{where}$$

where

$$\rho := \mu(1-\sqrt{1-2\kappa}), \quad \rho_3 := \mu(\sqrt{1+2\tilde{\gamma}/\mu}-1).$$

PROOF. The only thing missing for the application of theorem 2.1 is the Lipschitz constant  $L$  of the Lipschitz continuous derivative

$$DF : C \times C^n \rightarrow M_{n,n+1}. \text{ Evidently } Df(\alpha, a)(\delta, h) = (A - (\lambda + \alpha)I)h - \delta(x + a).$$

Hence

$$\| (Df(\alpha_2, a_2) - Df(\alpha_1, a_1))(\delta, h) \|_2 = \| (\alpha_1 - \alpha_2)h + \delta(a_1 - a_2) \|_2.$$

Since

$$\begin{aligned} \| (\alpha_1 - \alpha_2)h + \delta(a_2 - a_1) \|_2^2 &= |\alpha_1 - \alpha_2|^2 \|h\|_2^2 + |\delta|^2 \|a_2 - a_1\|_2^2 + \operatorname{Re}\{(\alpha_1 - \alpha_2)\bar{\delta}(h, a_2 - a_1)\} \\ &\leq |\alpha_1 - \alpha_2|^2 \|h\|_2^2 + |\delta|^2 \|a_2 - a_1\|_2^2 + 2|\alpha_1 - \alpha_2| |\delta| \|a_2 - a_1\|_2^2 \\ &\leq |\alpha_1 - \alpha_2|^2 \|h\|_2^2 + |\delta|^2 \|a_2 - a_1\|_2^2 + |\alpha_1 - \alpha_2|^2 |\delta|^2 + \|a_2 - a_1\|_2^2 \|h\|_2^2 \\ &= (|\alpha_1 - \alpha_2|^2 + \|a_2 - a_1\|_2^2) (|\delta|^2 + \|h\|_2^2). \end{aligned}$$

With well chosen pair  $(\delta, h)$  the equality signs occur, consequently

$$\| Df(\alpha_2, a_2) - Df(\alpha_1, a_1) \|_2 \leq (|\alpha_2 - \alpha_1|^2 + \|a_2 - a_1\|_2^2)^{\frac{1}{2}}.$$

This result implies that  $L = 1$  is a Lipschitz constant for the derivative  $Df$ . Hence, with  $L = 1$ , theorem 2.1 immediately leads to this particular case.  $\square$

So far only we dealt about an analytical expression for the error bounds.

But above all the Newton-Kantorovich approach provides a possibility to improve the approximate eigenvalue solution  $(\lambda_0, x_0) := (\lambda, x)$ .

Numerical results, obtained for  $F_{11}$ , the Frank matrix of order eleven, illustrate the performance of modified Newton's method in  $(\ker P)^1$ :

$$(\delta_k, h_k) = (\delta_{k-1}, h_{k-1}) - P^+ f(\delta_{k-1}, h_{k-1}), \quad k \in \mathbb{N}. \quad (4.4)$$

The general form of the Frank matrices is adequately illustrated by  $F_5$

which is

$$\begin{pmatrix} 5 & 4 & 3 & 2 & 1 \\ 4 & 4 & 3 & 2 & 1 \\ 0 & 3 & 3 & 2 & 1 \\ 0 & 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

"It can be shown that if  $\lambda_i$  is an eigenvalue then  $\lambda_i^{-1}$  is also an eigenvalue so that  $F_{2n+1}$  has one eigenvalue equal to unity. In general the larger eigenvalues are well-conditioned while the smaller ones are quite ill-conditioned." [16].

We used a single-precision approximation  $(\lambda_0, x_0)$  of the eigenpair for the smallest eigenvalue  $\tilde{\lambda} = 0.034\ 625\ 161\ 711$  where  $\lambda_0 = 0.034\ 623$ . This  $(\lambda_0, x_0)$  was obtained by a NAG-implementation of the QR algorithm.

In our implementation of the Newton-algorithm (4.4) we used double length precision: long reals. With the usual inner product in  $C^n$ , the Newton correction  $P^+ f(\delta_{k-1}, h_{k-1})$  was computed in the following way.

Firstly, the QR-decomposition is performed of the  $(n+1) \times n$  matrix  $P^*$  with Householder reflections:

$$P^* = \begin{pmatrix} (A - \lambda I)^* \\ x^* \end{pmatrix} = QR.$$

Then

$$P^+ = P^* (P P^*)^{-1} = QR (R^* Q^* QR)^{-1} = Q (R^*)^{-1}.$$

Hence

$$(\delta_k, h_k) := (\delta_{k-1}, h_{k-1}) - Q(R^*)^{-1} f(\delta_{k-1}, f_{k-1}), \quad k \in \mathbb{N}. \quad (4.5)$$

So in each sweep the lower triangular system

$$R^* z_{k-1} = f(\delta_{k-1}, f_{k-1}), k \in \mathbb{N}$$

has to be solved.

In Table 1 we give the results for the sequence  $\{\lambda_k\}$  where  $\lambda_k = \lambda_0 + \delta_k$ .

Table 1										
k	$\lambda_k$									
0	0.034	623								
1	0.034	625	161	711	439	764	197	778	019	752
2	0.034	625	161	711	425	640	061	346	944	761
3	0.034	625	161	711	425	620	857	605	864	163
4	0.034	625	161	711	425	620	855	414	507	969
5	0.034	625	161	711	425	620	855	414	291	505
6	0.034	625	161	711	425	620	855	414	291	351
correct numbers:		6			13	16	20	24		27

These numbers reflect the quadratic improvement in the first sweep of the modified method of Newton and they are in accordance with the figures of Symon and Wilkinson. [16] Evidently, if starting with the same  $(\lambda, x)$  and using single precision accuracy, one needs accumulation of innerproducts in double precision as an adequate precaution for otherwise the theoretic-al results will be invalidated by rounding errors.

Up to now, we have formulated error bounds in the forward sense. But theorem 2.1 also can be used to obtain error bounds in the mixed sense: both a perturbation  $H$  of the given matrix  $A$  and a perturbation  $h$  of the computed eigenvector  $x$  (, or of the computed eigenpair  $(\lambda, x)$ ,) are utilized in order to fulfill the eigenvalue-equation  $(A+H)(x+h) = \lambda(x+h)$ . The mixed error analysis involves the need for an innerproduct in the linear space  $M_{nn}$  of complex  $n \times n$  matrices. Hence we define  $(A, B)_E := \sum_{\ell, m=1}^n A_{\ell m} \bar{B}_{\ell m}$ ,  $A, B \in M_{nn}$ ; this product induces the many times used Euclidean (Frobenius) norm  $\|A\|_E := (A, A)_E^{1/2}$ . In the product space  $M_{nn} \times \mathbb{C}^n$  we consider the innerproduct



$$((A, x), (B, y))_\alpha := (A, B)_E + \alpha(x, y)_2, (A, x), (B, y) \in M_{nn} \times C^n,$$

where  $\alpha > 0$ .

THEOREM 4.2. Let be given  $A \in M_{nn}$ ,  $x \in C^n \setminus \{0\}$ ,  $\lambda \in C^n$  and  $\alpha > 0$ . The mappings  $f, P : M_{nn} \times C^n \rightarrow C^n$  are defined by

$$\left\{ \begin{array}{l} f(H, h) := (A+H)(x+h) - \lambda(x+h) \\ P(H, h) := (A-\lambda I)h + Hx. \end{array} \right. \quad (4.6)$$

$$(H, h) \in M_{nn} \times C^n, \quad (4.7)$$

With respect to the innerproducts  $(\cdot, \cdot)_\alpha$  in  $M_{nn} \times C^n$  and the usual Euclidean innerproduct in  $C^n$

$$PP^* = \|x\|_2^2 I + \frac{1}{\alpha}(A-\lambda I)(A-\lambda I)^*. \quad (4.8)$$

Df satisfies a Lipschitz condition with Lipschitz constant  $L = \frac{2}{\alpha}$ . If

- (i)  $\|(PP^*)^{-1}\|_2^{\frac{1}{2}} = \mu_n^{-1} \leq \mu^{-1}$ ,  $\mu_n$  being the smallest singular value of P;
- (ii)  $(r^*(PP^*)^{-1}r)^{\frac{1}{2}} = \tilde{\gamma} \leq \gamma$ , where  $r := f(0,0) = Ax - \lambda x$ ;
- (iii)  $\kappa := \gamma \mu^{-1} L < \frac{1}{2}$ ,

then the distance  $\hat{d}$  of  $(0,0)$  to  $\hat{S} := \{(H, h) | f(H, h) = 0\}$  satisfies the inequalities

$$\rho_1 := \rho_3 \left(1 + \frac{L^2 \rho_3^2}{(\mu - L\rho_3)^2}\right)^{-\frac{1}{2}} < \hat{d} < \rho$$

where

$$\rho = \mu(1 - \sqrt{1 - 2\kappa})/L, \quad \rho_3 = \mu(\sqrt{1 + 2L\tilde{\gamma}\mu^{-1}} - 1)/L.$$

PROOF. With respect to the innerproducts  $(\cdot, \cdot)_\alpha$  in  $M_{nn} \times C^n$  and  $(\cdot, \cdot)_2$  in  $C^n$



we derive for the adjoint  $P^*: C^n \rightarrow M_{nn} \times C^n$  of  $P = Df(0,0)$ :

$$\begin{aligned} ((H,h), P^*k)_\alpha &= (P(H,h), k)_2 = ((A-\lambda I)h + Hx, k)_2 \\ &= (h, (A-\lambda I)^*k)_2 + (H, kx^*)_E = ((H,h), (kx^*, \frac{1}{\alpha}(A-\lambda I)^*k)_\alpha. \end{aligned}$$

So for each  $k \in C^n$  we have

$$P^*k = (kx^*, \frac{1}{\alpha}(A-\lambda I)^*k). \quad (4.9)$$

With (4.7) we find for  $PP^*: C^n \rightarrow C^n$ :

$$PP^*k = (\frac{1}{\alpha}(A-\lambda I)(A-\lambda I)^* + \|x\|_2^2 I)k.$$

The Hermitian matrix  $PP^*$  is positive definite, thus  $P$  is surjective. Now, we compute a Lipschitz constant for  $Df$ . Evidently,  $Df(E,e)(H,h) = (A+E-\lambda I)h + H(x+e)$ . Thus,

$$\begin{aligned} \|Df(E_1, e_1) - Df(E_2, e_2)(H, h)\|_2^2 &= \|(E_1 - E_2)h + H(e_1 - e_2)\|_2^2 \\ &\leq 2\|H\|_2^2 \|e_1 - e_2\|_2^2 + 2\|E_1 - E_2\|_2^2 \|h\|_2^2 \\ &\leq \frac{2}{\alpha}(\|E_1 - E_2\|_2^2 + \alpha\|e_1 - e_2\|_2^2)(\|H\|_2^2 + \alpha\|h\|_2^2) \\ &\leq \frac{2}{\alpha} \|(E_1 - E_2, e_1 - e_2)\|_\alpha^2 \cdot \|(H, h)\|_\alpha^2. \end{aligned}$$

Consequently,

$$\|Df(E_1, e_1) - Df(E_2, e_2)\| \leq \sqrt{\frac{2}{\alpha}} \|(E_1 - E_2, e_1 - e_2)\|_\alpha.$$

So for the Lipschitz constant  $L$  of derivative  $Df$  we have found  $L \leq \sqrt{\frac{2}{\alpha}}$ . With this bound for  $L$  we are able to apply theorem 2.1; that gives the upper bound  $\rho$  and the lower bound  $\rho_1$  for the necessary perturbation  $(H, h)$  such that  $f(H, h) = 0$ .  $\square$

This application of the minimization process in the normal space of a level surface of  $f$  at  $(0,0)$  can be applied for the construction of vector  $x+h$  nearby to  $x$  and a matrix  $A+H$ , nearby to  $A$ , with a prescribed eigen-

value. A small weight  $\alpha$  in the innerproduct  $(\cdot, \cdot)_\alpha$  effects in the minimization result the term  $\|H\|_E^2$  in  $\|(H, h)\|_\alpha^2$  to be small relatively  $\|h\|_2^2$ .

In our experiments the modified method of Newton, in formula

$$(H_k, h_k) := (H_{k-1}, h_{k-1}) - P^*(PP^*)^{-1} f(H_{k-1}, h_{k-1}), \quad k \in \mathbb{N},$$

with starting value  $(H_0, h_0) = (0, 0) \in M_{nn} \times C^n$ , has been implemented in the following way.

Rowwise the matrix  $H_k$  is set in a linear  $n^2$  array. So the  $n \times (n^2 + n)$  matrix  $P$  equals the Kroneckerproduct  $I \otimes x^T$ , bordered by  $A - \lambda I$ . Then the matrix representation of the adjoint  $P^*$  of  $P$  ( $(\cdot, \cdot)_\alpha$ ) equals, as follows from (4.9),

$$\begin{pmatrix} I \otimes \bar{x} \\ \dots\dots\dots \\ \alpha^{-1} \overline{A - \lambda I}^T \end{pmatrix}$$

The Newton correction  $P^*(PP^*)^{-1} f(H_{k-1}, h_{k-1})$  has been obtained from the QR-decomposition of the appropriately adapted  $P^*$ :

$$QR = \begin{pmatrix} I \otimes \bar{x} \\ \dots\dots\dots \\ \alpha^{-1/2} \overline{A - \lambda I}^T \end{pmatrix}$$

Then

$$PP^* = \bar{R}^T \bar{Q}^T QR.$$

Consequently,

$$(H_k, h_k) = (H_{k-1}, h_{k-1}) - \bar{Q}\bar{R}^{-T} f(H_{k-1}, h_{k-1}), \quad k \in \mathbb{N}.$$

So, after the initial QR-decomposition, in each step one has to solve a lower triangular linear system with  $n$  unknowns.

As a test we used this algorithm on the Frank matrix  $F$  of order 11,  $(\lambda, x)$  being, as in the preceding example, a single-precision approximation of the eigenpair corresponding with the smallest eigenvalue of  $F$ . Single

precision words on ICL 1902 are of 14 hexadecimals and double-precision words (long reals) are of 28 hexadecimals, these correspond to 16.8 and 33.7 decimal digits respectively. In this example we have chosen  $\alpha = 10^{-12}$ . The smallest eigenvalue is, as mentioned above, ill-conditioned, so  $\lambda$  is fairly inaccurate, but  $f(H_0, h_0) = x = Ax - \lambda x$  was of the order of the noise level of the computer, i.e.  $2^{-56}$ .

In table 2 and 3 we give an informative selection of the numerical results.

Table 2	
$\ f(H_3, h_3)\ $	$= 5.099 \times 10^{-33}$
$\ H_3\ _E$	$= 1.423 \times 10^{-12}$
$\ h_3\ _2$	$= 3.562 \times 10^{-6}$
$\ (H_3, h_3)\ _{10^{-12}}$	$= 3.835 \times 10^{-12}$

Table 3	
(i,j)	$(H_3)_{ij}$
(2,11)	$7.151 \times 10^{-13}$
(2,10)	$-6.904 \times 10^{-13}$
(1,11)	$-6.522 \times 10^{-13}$
(1,10)	$6.296 \times 10^{-13}$
(2,9)	$3.209 \times 10^{-13}$
(1,9)	$-2.926 \times 10^{-13}$
(2,8)	$-0.953 \times 10^{-13}$
(1,8)	$0.869 \times 10^{-13}$
(3,11)	$-0.672 \times 10^{-13}$
(3,10)	$0.649 \times 10^{-13}$
(8,1)	$2.241 \times 10^{-27}!$

Extreme elements of  $H_3$

The smallness of this feasible perturbation  $H_3$  bases on the ill-condition of the smallest eigenvalue. As the individual condition number  $10^6$  effects that a perturbation of order  $10^{-12}$  results in an error of order  $10^{-6}$  [19], here we see that a forced error of order  $10^{-6}$  can be explained by a perturbation of order  $10^{-12}$ .

The described method can be generalized by construct a matrix  $A+H$ , nearby to a given matrix  $A$ , such that the spectrum of  $A+H$  differs in a prescribed way from that of  $A$ . In a next memorandum, we report about that problem.

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